647510 **MSC1MatC102x** **Seat No :________**

M.Sc.(Maths.) Semester - 1 (*CBCS*) Examination

Nov/Dec - 2022 [NEW COURSE]

Real Analysis(Core (New))

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1 (a) Answer any one question out of two. **4 Marks**

(1) Let C be a collection of subsets of X. Show that there is a smallest σ-algebra of subsets of X containing C. 4

(2) Show that finite union of measurable sets is measurable. 4

Q.1 (b) Answer any two questions out of three. **10 Marks**

(1) Show that the outer measure of an interval is its length. 5

(2) Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$. Suppose that given $\epsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \epsilon$. Show that E is measurable. 5

(3) Let f be a measurable simple function defined on a measurable set E. Show that for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E such that $\mathrm{f(x) = g(x)}$ for all $\mathrm{x \in F}$ and $\mathrm{m(E - F) < \epsilon}$. 5

Q.2 (a) Answer any one question out of two. **4 Marks**

(1) Let (f_n) be a sequence of measurable functions defined on a set of finite measure E, and let $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and for all $x \in E$, where M is a constant. If $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$, then show that $\int_E f = \lim\limits_{n\to\infty} \int_E f_n$. 4

(2) Let f be a nonnegative measurable function defined over a measurable set E. If $\int_E f = 0$, then show that $f = 0$ a.e. on E. 4

Q.2 (b) Answer any two questions out of three. **10 Marks**

(1) If ϕ and ψ are measurable simple functions vanishing outside a set of finite measure and if $a, b \in \mathbb{R}$, then show that $\int (a\phi + b\psi) = a\int \phi + b\int \psi$. 5

(2) Let f be nonnegative integrable function over a measurable set E. If $\epsilon > 0$ is given, then show that there is $\delta > 0$ such that $\int_A f < \epsilon$ whenever A is a measurable subset of E with $m(A) < \delta$. 5

(3) Show that convergence in measure may not imply pointwise convergence. 5

Q.3 (a) Answer any one question out of two. **4 Marks**

(1) Let f be defined on $[0,1]$ by $f(0) = 0$ and $f(x) = x \sin\frac{1}{x}$ if $x \neq 0$. Show that f is not of bounded variation on $[0,1]$. 4

(2) If f is integrable on $[a, b]$ and if $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$, then show that $f = 0$ a.e. on $[a, b]$. 4

Q.3 (b) Answer any two questions out of three. **10 Marks**

(1) In usual notations prove that if f is of bounded variation on $[a, b]$, then $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$. 5

(2) Let X and Y be Banach spaces and $T: X \to Y$ a closed linear operator. Show that the operator T is bounded. 5

(3) Show that a function F on $[a, b]$ is an indefinite integral if and only if it is absolutely continuous on $[a, b]$.. 5

Q.4 (a) Answer any one question out of two. **4 Marks**

(1) Show that the product of two square integrable functions is integrable. 4

(2) If f and g are essentially bounded functions, then show that $f + g$ is essentially bounded and $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. 4

Q.4 (b) Answer any two questions out of three. **10 Marks**

(1) Let $1 \leq p \leq \infty$. If $f, g \in L^p$, then show that $f + g \in L^p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. 5

(2) If $1 \leq p < \infty$, then show that L^p is complete. 5

(3) Show that a normed linear space X is complete if and only if every absolutely summable series is summable. 5

Q.5 (a) Answer any one question out of two. **4 Marks**

(1) If f is integrable over a measurable set E, then show that $\left|\int_E f\right| \leq \int_E |f|$. 4

(2) Give an example of a function defined on $[0,1]$ which is Lebesgue integrable but not Riemann integrable. 4

Q.5 (b) Answer any two questions out of three. **10 Marks**

(1) If (f_n) is a sequence of nonnegative measurable functions defined on a measurable set E and if $f_n \to f$ pointwise a.e. on E, then show that $\int_E f \leq \liminf_n \int_E f_n$. 5

(2) If (E_n) is an increasing sequence of measurable sets and if $E = \bigcup_n E_n$, then show that $m(E) = \lim_{n\to\infty} m(E_n)$. 5

(3) If f is absolutely continuous on $[a, b]$, then show that f is of bounded variation on $[a, b]$. 5

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